

Two-Dimensional Spaces, Volume 1  
**GEOMETRY OF LENGTHS,  
AREAS, AND VOLUMES**

*James W. Cannon*



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AMERICAN MATHEMATICAL SOCIETY  
Providence, Rhode Island

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Two-Dimensional Spaces, Volume 2

# TOPOLOGY AS FLUID GEOMETRY

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Two-Dimensional Spaces, Volume 3

# NON-EUCLIDEAN GEOMETRY AND CURVATURE

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