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# Separable Type Representations of Matrices and Fast Algorithms

Volume 1

Basics. Completion Problems.

Multiplication and Inversion Algorithms

# Contents

Preface . . . . .	xi
Introduction . . . . .	xiii
 <b>Part I Basics on Separable, Semi-separable and Quasiseparable Representations of Matrices</b>	
 <b>1 Matrices with Separable Representation and Low Complexity Algorithms</b>	
§1.1 Rank and related factorizations . . . . .	4
§1.2 Definitions and first examples . . . . .	6
§1.3 The algorithm of multiplication by a vector . . . . .	8
§1.3.1 Forward and backward computation of $y$ . . . . .	8
§1.3.2 Forward-backward computation of $y$ . . . . .	10
§1.4 Systems with homogeneous boundary conditions associated with matrices in diagonal plus separable form . . . . .	12
§1.4.1 Forward and backward systems . . . . .	12
§1.4.2 Forward-backward descriptor systems . . . . .	15
§1.5 Multiplication of matrices . . . . .	16
§1.5.1 Product of matrices with separable representations . . . . .	16
§1.5.2 Product of matrices with diagonal plus separable representations . . . . .	18
§1.6 Schur factorization and inversion of block matrices . . . . .	21
§1.7 A general inversion formula . . . . .	25
§1.8 Inversion of matrices with diagonal plus separable representation . . . . .	27
§1.9 LDU factorization of matrices with diagonal plus separable representation . . . . .	31
§1.10 Solution of linear systems in the presence of the $LDU$ factorization of the matrix of the system in diagonal plus separable form . . . . .	39
§1.11 Comments . . . . .	44

<b>2</b>	<b>The Minimal Rank Completion Problem</b>	
§2.1	The definition. The case of a $2 \times 2$ block matrix . . . . .	45
§2.2	Solution of the general minimal rank completion problem. Examples . . . . .	51
§2.3	Uniqueness of the minimal rank completion . . . . .	61
§2.4	Comments . . . . .	66
<b>3</b>	<b>Matrices in Diagonal Plus Semiseparable Form</b>	
§3.1	Definitions . . . . .	67
§3.2	Semiseparable order and minimal semiseparable generators . . . .	69
§3.3	Comments . . . . .	73
<b>4</b>	<b>Quasiseparable Representations: The Basics</b>	
§4.1	The rank numbers and quasiseparable order. Examples . . . . .	75
§4.1.1	The definitions . . . . .	75
§4.1.2	The companion matrix . . . . .	76
§4.1.3	The block companion matrix . . . . .	76
§4.1.4	Tridiagonal matrices and band matrices . . . . .	77
§4.1.5	Matrices with diagonal plus semiseparable representations . . . . .	78
§4.2	Quasiseparable generators . . . . .	79
§4.3	Minimal completion rank, rank numbers, and quasiseparable order . . . . .	82
§4.4	The quasiseparable and semiseparable generators . . . . .	83
§4.5	Comments . . . . .	84
<b>5</b>	<b>Quasiseparable Generators</b>	
§5.1	Auxiliary matrices and relations . . . . .	86
§5.2	Existence and minimality of quasiseparable generators . . . . .	89
§5.3	Examples . . . . .	92
§5.4	Quasiseparable generators of block companion matrices viewed as scalar matrices . . . . .	99
§5.5	Minimality conditions . . . . .	102
§5.6	Sets of generators. Minimality and similarity . . . . .	105
§5.7	Reduction to minimal quasiseparable generators . . . . .	110
§5.8	Normal quasiseparable generators . . . . .	112
§5.9	Approximation by matrices with quasiseparable representation . . . . .	115
§5.10	Comments . . . . .	117

**6 Rank Numbers of Pairs of Mutually Inverse Matrices, Asplund Theorems**

§6.1 Rank numbers of pairs of inverse matrices . . . . . 120

§6.2 Rank numbers relative to the main diagonal.  
Quasiseparable orders . . . . . 122

§6.3 Green and band matrices . . . . . 123

§6.4 The inverses of diagonal plus Green of order one matrices . . . . . 126

§6.5 Minimal completion ranks of pairs of mutually inverse  
matrices. The inverse of an irreducible band matrix . . . . . 129

§6.6 Linear-fractional transformations of matrices . . . . . 134

§6.6.1 The definition and the basic property . . . . . 134

§6.6.2 Linear-fractional transformations of Green and  
band matrices . . . . . 135

§6.6.3 Unitary Hessenberg and Hermitian matrices . . . . . 135

§6.6.4 Linear-fractional transformations of diagonal plus  
Green of order one matrices . . . . . 136

§6.7 Comments . . . . . 137

**7 Unitary Matrices with Quasiseparable Representations**

§7.1 QR and related factorizations of matrices . . . . . 139

§7.2 The rank numbers and quasiseparable generators . . . . . 142

§7.3 Factorization representations . . . . . 143

§7.3.1 Block triangular matrices . . . . . 143

§7.3.2 Factorization of general unitary matrices and  
compression of generators . . . . . 146

§7.3.3 Generators via factorization . . . . . 151

§7.4 Unitary Hessenberg matrices . . . . . 155

§7.5 Efficient generators . . . . . 157

§7.6 Comments . . . . . 161

**Part II Completion of Matrices with Specified Bands**

**8 Completion to Green Matrices**

§8.1 Auxiliary relations . . . . . 165

§8.2 Completion formulas . . . . . 167

§8.3 Comments . . . . . 177

<b>9 Completion to Matrices with Band Inverses and with Minimal Ranks</b>	
§9.1 Completion to invertible matrices . . . . .	180
§9.2 The LDU factorization . . . . .	182
§9.3 The Permanence Principle . . . . .	186
§9.4 The inversion formula . . . . .	191
§9.5 Completion to matrices of minimal ranks . . . . .	197
§9.6 Comments . . . . .	199
 <b>10 Completion of Special Types of Matrices</b>	
§10.1 The positive case . . . . .	201
§10.2 The Toeplitz case . . . . .	206
§10.3 Completion of specified tridiagonal parts with identities on the main diagonal . . . . .	208
§10.3.1 The general case . . . . .	208
§10.3.2 The Toeplitz case . . . . .	210
§10.4 Completion of special $2 \times 2$ block matrices . . . . .	211
§10.4.1 Completion formulas . . . . .	211
§10.4.2 Completion to invertible and positive matrices . . . . .	214
§10.4.3 Completion to matrices of minimal ranks . . . . .	215
§10.5 Comments . . . . .	217
 <b>11 Completion of Mutually Inverse Matrices</b>	
§11.1 The statement and preliminaries . . . . .	219
§11.2 The basic theorem . . . . .	221
§11.3 The direct method . . . . .	225
§11.4 The factorization . . . . .	227
§11.5 Comments . . . . .	228
 <b>12 Completion to Unitary Matrices</b>	
§12.1 Auxiliary relations . . . . .	229
§12.2 An existence and uniqueness theorem . . . . .	230
§12.3 Unitary completion via quasiseparable representation . . . . .	237
§12.3.1 Existence theorem . . . . .	237
§12.3.2 Diagonal correction for scalar matrices . . . . .	242
§12.4 Comments . . . . .	244

**Part III    Quasiseparable Representations of Matrices,  
Descriptor Systems with Boundary Conditions  
and First Applications**

**13 Quasiseparable Representations and Descriptor Systems  
with Boundary Conditions**

§13.1 The algorithm of multiplication by a vector . . . . . 247

§13.2 Descriptor systems with homogeneous boundary conditions . . . . 250

§13.3 Examples . . . . . 253

§13.4 Inversion of triangular matrices . . . . . 255

§13.5 Comments . . . . . 260

**14 The First Inversion Algorithms**

§14.1 Inversion of matrices in quasiseparable representation with  
invertible diagonal elements . . . . . 261

§14.2 The extension method for matrices with quasiseparable/semi-  
separable representations . . . . . 271

    §14.2.1 The inversion formula . . . . . 272

    §14.2.2 The orthogonalization procedure . . . . . 275

§14.3 Comments . . . . . 278

**15 Inversion of Matrices in Diagonal Plus Semiseparable Form**

§15.1 The modified inversion formula . . . . . 279

§15.2 Scalar matrices with diagonal plus semiseparable  
representation . . . . . 284

§15.3 Comments . . . . . 293

**16 Quasiseparable/Semiseparable Representations and One-direction Systems**

§16.1 Systems with diagonal main coefficients and homogeneous  
boundary conditions . . . . . 295

§16.2 The general one-direction systems . . . . . 301

§16.3 Inversion of matrices with quasiseparable/semiseparable  
representations via one-direction systems . . . . . 305

§16.4 Comments . . . . . 307

**17 Multiplication of Matrices**

§17.1 The rank numbers of the product . . . . . 309

§17.2 Multiplication of triangular matrices . . . . . 310

§17.3 The general case . . . . . 316

§17.4 Multiplication by triangular matrices . . . . . 319

§17.5 Complexity analysis . . . . . 322

§17.6 Product of matrices with semiseparable representations . . . . . 324

§17.7 Comments . . . . . 326

**Part IV Factorization and Inversion**

**18 The LDU Factorization and Inversion**

§18.1 Rank numbers and minimal completion ranks . . . . . 329

§18.2 The factorization algorithm . . . . . 330

§18.3 Solution of linear systems and analog of Levinson algorithm . . . 341

§18.4 The inversion formula . . . . . 347

§18.5 The case of a diagonal plus semiseparable representation . . . . . 349

§18.6 Comments . . . . . 351

**19 Scalar Matrices with Quasiseparable Order One**

§19.1 Inversion formula . . . . . 353

§19.2 Examples . . . . . 360

§19.3 Inversion algorithm with scaling . . . . . 366

§19.4 The case of diagonal plus semiseparable representation . . . . . 369

§19.5 The case of a tridiagonal matrix . . . . . 370

§19.6 Comments . . . . . 372

**20 The QR-Factorization Based Method**

§20.1 Factorization of triangular matrices . . . . . 373

§20.2 The first factorization theorem . . . . . 377

§20.3 The second factorization theorem . . . . . 380

§20.4 Solution of linear systems . . . . . 384

§20.5 Complexity . . . . . 385

§20.6 The case of scalar matrices . . . . . 387

§20.7 Comments . . . . . 390

**Bibliography . . . . . 393**

**Index . . . . . 397**