

Herbert Koch
Daniel Tataru
Monica Vişan

Dispersive Equations and Nonlinear Waves

Generalized Korteweg–de Vries, Nonlinear
Schrödinger, Wave and Schrödinger Maps



Birkhäuser

Contents

Preface	xi
Nonlinear Dispersive Equations	1
<i>Herbert Koch</i>	
1 Introduction	3
2 Stationary phase and dispersive estimates	5
2.1 Examples and dispersive estimates	13
2.1.1 The Schrödinger equation	13
2.1.2 The Airy function and the Airy equation	13
2.1.3 Laplacian and related operators	16
2.1.4 Gaussians, heat and Schrödinger equation	17
2.1.5 The half-wave equation	18
2.1.6 The Klein-Gordon half wave	20
2.1.7 The Kadomtsev–Petviashvili equation	20
3 Strichartz estimates and small data for the nonlinear Schrödinger equation	23
3.1 Strichartz estimates for the Schrödinger equation	23
3.2 Strichartz estimates for the Airy equation	26
3.3 The Kadomtsev–Petviashvili equation	27
3.4 The (half-) wave equation and the Klein–Gordon equation	28
3.5 The endpoint Strichartz estimate	28
3.6 Small data solutions to the nonlinear Schrödinger equation	32
3.7 Initial data in L^2	33
3.8 Initial data in \dot{H}^1 for $d \geq 3$	37
3.9 Initial data in $H^1(\mathbb{R}^d)$	39

4	Functions of bounded p-variation	41
4.1	Functions of bounded p -variation and the spaces U^p and V^p	43
4.2	Duality and the Riemann–Stieltjes integral	49
4.3	Step functions are dense	52
4.4	Convolution and regularization	53
4.5	More duality	56
4.6	Consequences of Minkowski’s inequality	59
4.7	The bilinear form as integral	60
4.8	Differential equations with rough paths	62
4.9	The Brownian motion	64
4.10	Adapted function spaces	65
4.10.1	Strichartz estimates	68
4.10.2	Estimates by duality	70
4.10.3	High modulation estimates	71
5	Convolution of measures on hypersurfaces, bilinear estimates, and local smoothing	73
6	Well-posedness for nonlinear dispersive equations	87
6.1	Adapted function spaces approach for a model problem	87
6.2	The (generalized) KdV equation	89
6.3	The derivative nonlinear Schrödinger equation	103
6.4	The Kadomtsev–Petviashvili II equation	106
7	Appendix A: Young’s inequality and interpolation	111
7.1	Complex interpolation: The Riesz–Thorin theorem	119
8	Appendix B: Bessel functions	123
9	Appendix C: The Fourier transform	127
9.1	The Fourier transform in L^1	127
9.2	The Fourier transform of Schwartz functions	128
9.3	Tempered distributions	130
	Bibliography	135

Geometric Dispersive Evolutions**139***Daniel Tataru*

1	Introduction	141
2	Maps into manifolds	143
2.1	The tangent bundle and covariant differentiation	143
2.2	Special targets	145
2.3	Sobolev spaces	145
2.4	S^2 and targets: homotopy classes and equivariance	147
2.5	Frames and gauge freedom	148
3	Geometric pde's	151
3.1	Harmonic maps	151
3.2	The harmonic heat flow	153
3.3	Wave maps	155
3.4	Schrödinger maps	158
4	Wave maps	161
4.1	Small data heuristics	161
4.2	A perturbative set-up	161
4.2.1	The Strichartz norms	162
4.2.2	The null structure	162
4.2.3	The null frame spaces	165
4.2.4	The paradifferential equation and renormalization	167
4.3	Function spaces	168
4.3.1	Frequency envelopes	170
4.3.2	Linear analysis in the S and N spaces	171
4.3.3	Multilinear estimates	172
4.4	Renormalization	174
4.5	The small data result	177
4.5.1	The a priori estimate	177
4.5.2	Global existence and regularity	178
4.5.3	Weak Lipschitz dependence on the initial data	179
4.5.4	Rough solutions and continuous dependence on the initial data	179
4.6	Energy dispersion	179
4.6.1	Energy dispersion and multilinear estimates	181
4.6.2	Compare the initial data of ϕ and $\tilde{\phi}$	183
4.6.3	Compare the low frequencies of ϕ and $\tilde{\phi}$	183
4.6.4	Compare the high frequencies	183
4.7	Energy and Morawetz estimates	184
4.7.1	Notations	185

4.7.2	The energy-momentum tensor	185
4.7.3	Energy estimates	186
4.7.4	The energy of self-similar maps	187
4.7.5	Morawetz estimates	188
4.8	The threshold theorem	192
4.9	Further developments	197
5	Schrödinger maps	201
5.1	Frames and gauges	201
5.2	Function spaces	204
5.3	The small data result	210
5.3.1	Bounds for the harmonic heat flow	210
5.3.2	Bounds for the Schrödinger map flow	212
5.3.3	Rough solutions and continuous dependence.	212
5.4	Further developments	214
5.4.1	Other targets	214
5.4.2	Large data	215
5.4.3	Near soliton behavior	216
	Bibliography	219
	Dispersive Equations	223
	<i>Monica Vişan</i>	
1	Notation	225
2	Dispersive and Strichartz estimates	227
2.1	The linear Schrödinger equation	227
2.2	The Airy equation	229
2.3	The linear wave equation	230
2.4	From dispersive to Strichartz estimates	232
2.5	Bilinear Strichartz and local smoothing estimates	236
3	An inverse Strichartz inequality	239
4	A linear profile decomposition	245
5	Stability theory for the energy-critical NLS	251
6	A large data critical problem	259
7	A Palais–Smale type condition	261
8	Existence of minimal blowup solutions and their properties	271

9	Long-time Strichartz estimates and applications	281
9.1	A long-time Strichartz inequality	281
9.2	The rapid frequency cascade scenario	287
10	Frequency-localized interaction Morawetz inequalities and applications	291
10.1	A frequency-localized interaction Morawetz inequality	294
10.2	The quasi-soliton scenario	300
11	Appendix A: Background material	303
11.1	Compactness in L^p	303
11.2	Littlewood–Paley theory	304
11.3	Fractional calculus	305
11.4	A paraproduct estimate	306
	Bibliography	309