Peter K. Friz · Martin Hairer

A Course on Rough Paths

With an Introduction to Regularity Structures

Springer

Contents

1 Introduction 1
1.1 Controlled differential equations 1
1.2 Analogies with other branches of mathematics 6
1.3 Regularity structures 8
1.4 Frequently used notations 9
1.5 Rough path theory works in infinite dimensions 11

2 The space of rough paths 13
2.1 Basic definitions 13
2.2 The space of geometric rough paths 16
2.3 Rough paths as Lie-group valued paths 17
2.4 Geometric rough paths of low regularity 19
2.5 Exercises 20
2.6 Comments 25

3 Brownian motion as a rough path 27
3.1 Kolmogorov criterion for rough paths 27
3.2 Itô Brownian motion 31
3.3 Stratonovich Brownian motion 32
3.4 Brownian motion in a magnetic field 34
3.5 Cubature on Wiener Space 39
3.6 Scaling limits of random walks 40
3.7 Exercises 42
3.8 Comments 46

4 Integration against rough paths 47
4.1 Introduction 47
4.2 Integration of 1-forms 48
4.3 Integration of controlled rough paths 55
4.4 Stability I: rough integration 60
4.5 Controlled rough paths of lower regularity 61

4.6 Exercises 63
4.7 Comments 66

5 Stochastic integration and Itô's formula 67
5.1 Itô integration 67
5.2 Stratonovich integration 69
5.3 Itô's formula and Föllmer 70
5.4 Backward integration 75
5.5 Exercises 78
5.6 Comments 82

6 Doob–Meyer type decomposition for rough paths 83
6.1 Motivation from stochastic analysis 83
6.2 Uniqueness of the Gubinelli derivative and Doob–Meyer 85
6.3 Brownian motion is truly rough 87
6.4 A deterministic Norris' lemma 88
6.5 Brownian motion is Hölder rough 90
6.6 Exercises 93
6.7 Comments 93

7 Operations on controlled rough paths 95
7.1 Relation between rough paths and controlled rough paths 95
7.2 Lifting of regular paths. 96
7.3 Composition with regular functions. 97
7.4 Stability II: Regular functions of controlled rough paths 98
7.5 Itô's formula revisited 100
7.6 Controlled rough paths of low regularity 101
7.7 Exercises 102

8 Solutions to rough differential equations 105
8.1 Introduction 105
8.2 Review of the Young case: a priori estimates 106
8.3 Review of the Young case: Picard iteration 107
8.4 Rough differential equations: a priori estimates 109
8.5 Rough differential equations 112
8.6 Stability III: Continuity of the Itô–Lyons map 116
8.7 Davie's definition and numerical schemes 117
8.8 Lyons' original definition 119
8.9 Stability IV: Flows 120
8.10 Exercises 121
8.11 Comments 122

9 Stochastic differential equations 123
9.1 Itô and Stratonovich equations 123
9.2 The Wong–Zakai theorem 124
9.3 Support theorem and large deviations 125
9.4 Exercises 126
9.5 Comments 127

10 Gaussian rough paths 129
10.1 A simple criterion for Hölder regularity 129
10.2 Stochastic integration and variation regularity of the covariance 131
10.3 Fractional Brownian motion and beyond 139
10.4 Exercises 142
10.5 Comments 147

11 Cameron–Martin regularity and applications 149
11.1 Complementary Young regularity 149
11.2 Concentration of measure 154
11.2.1 Borell's inequality 154
11.2.2 Fernique theorem for Gaussian rough paths 155
11.2.3 Integrability of rough integrals and related topics 156
11.3 Malliavin calculus for rough differential equations 160
11.3.1 Bouleau–Hirsch criterion and Hörmander's theorem 160
11.3.2 Calculus of variations for ODEs and RDEs 161
11.3.3 Hörmander's theorem for Gaussian RDEs 164
11.4 Exercises 166
11.5 Comments 168

12 Stochastic partial differential equations 169
12.1 Rough partial differential equations 169
12.1.1 Linear theory: Feynman–Kac 169
12.1.2 Nonlinear theory: flow transformation method 173
12.1.3 Rough viscosity solutions 178
12.2 Stochastic heat equation as a rough path 180
12.2.1 The linear stochastic heat equation 182
12.3 Exercises 186
12.4 Comments 190

13 Introduction to regularity structures 191
13.1 Introduction 191
13.2 Definition of a regularity structure and first examples 192
13.2.1 The canonical polynomial structure 194
13.2.2 The rough path structure 195
13.3 Definition of a model and first examples 197
13.3.1 The canonical polynomial model 200
13.3.2 The rough path model 202
13.4 Wavelets and the reconstruction theorem 204

13.5 Exercises 209
13.6 Comments 210

14 Operations on modelled distributions 211
14.1 Differentiation 211
14.2 Products and composition by regular functions 212
14.3 Schauder estimates and admissible models 215
14.4 Exercises 219

15 Application to the KPZ equation 221
15.1 Formulation of the main result 221
15.2 Construction of the associated regularity structure 224
15.3 The structure group 227
15.4 Canonical lifts of regular functions 229
15.5 Renormalisation of the KPZ equation 230
15.5.1 The renormalisation group 230
15.5.2 The renormalised equations 232
15.5.3 Convergence of the renormalised models 234
15.6 The KPZ equation and rough paths 238

References 241

Index 249